Scaling Limits of a Tagged Particle in the Exclusion Process with Variable Diffusion Coefficient

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Abstract We prove a law of large numbers and a central limit theorem for a tagged particle in a symmetric simple exclusion process in $\mathbb{Z}$ with variable diffusion coefficient. The scaling limits are obtained from a similar result for the current through $-1/2$ for a zero-range process with bond disorder. For the CLT, we prove convergence to a fractional Brownian motion of Hurst exponent $1/4$.

Keywords Tagged particle · Random environment · Zero-range process · Hydrodynamic limit · Fractional Brownian motion

1 Introduction

A classical problem in statistical mechanics consists in obtaining the scaling limit of a tagged (sometimes called tracer) particle in a system of interacting particles. In general, we expect the limiting process to be a Brownian motion for the system in equilibrium [16], or a time-inhomogeneous diffusion in general [14]. However, in the special case of a one-dimensional, nearest-neighbors system (the so-called *single-file diffusion*), a sub-diffusive behavior is expected [10]. In the case of a symmetric, simple exclusion process, the scaling limit of a tagged particle is given by $\sqrt{(1-\varphi)/\varphi}\,\mathcal{Z}_t$, where $\mathcal{Z}_t$ is a fractional Brownian motion of Hurst exponent $1/4$ and φ is the density of particles [2, 6].

In recent years, the evolution of a random walk in random environment has received a lot of attention. Therefore, to investigate the behavior of particle systems in random environments seems to be a natural step ahead. A simple example of a single-file diffusion with disorder is what we call the exclusion process with variable diffusion coefficient. In

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this model, to each particle we attach a different diffusion coefficient, and we let the system evolve in such a way that each particle performs a continuous-time, simple random walk with its own diffusion coefficient, conditioned to have at most one particle per site (the *exclusion* rule). For a physical interpretation of this model and applications, see [1, 3, 5].

In these notes we generalize the results of [2, 6] to the simple exclusion process with variable diffusion coefficient. We will see that the scaling limits depend on the diffusion coefficient only through an *effective* diffusion coefficient, obtained from the related homogenization problem. See the discussion in [5], where the case of a simple exclusion process with two different diffusion coefficients is studied in detail. Due to the variable diffusion coefficient, the approach of [2, 6] can not be repeated here, since in the associated stirring process, particles not only exchange their positions, but also their diffusion coefficients. We adopt here the approach of [17, 18], also included in [5]. There, scaling limits for the tagged particle are related to scaling limits for the current through the bond $\langle -1, 0 \rangle$ in a *zero-range* dynamics. We call this relation the *coupling method*. The main problem is that the associated zero-range dynamics in the case of variable diffusion coefficient is a particle system with *bond disorder*. As proposed by Rost and Vares [20], scaling limits for the current can be obtained from the hydrodynamic limit and fluctuations of the density of particles. To obtain those scaling limits in the presence of random media, various methods have been developed [7, 8, 19]. However, the method developed recently in [9] applies directly to our situation, avoiding the hard non-gradient tools needed in the other references. This is specially convenient in this setting, since the so-called spectral gap for the zero-range process is not uniform in the density for our model. In fact, we think of this work as a nice direct application of the results in [9].

This article is organized as follows. In Sect. 2 we define the model and state the main results. In Sect. 3 we construct the associated zero-range dynamics and we prove the main results. In Sect. 4 we discuss the relevance of the assumptions made on initial conditions and on the variable diffusion coefficients. Although the flavour of this paper is rather mathematical, some of the proofs are only sketched, to make the exposition accessible to a broader audience. We hope that the interested reader can easily fill the gaps in proofs by following our guidelines.

2 Definitions and Results

Consider a system of continuous-time, interacting random walks $\{x_i(t); i \in \mathbb{Z}\}$ on the integer lattice $\mathbb{Z}$, and a sequence of positive numbers $\{\lambda_i; i \in \mathbb{Z}\}$. The dynamics of this system is the following. The particle $x_i(t)$ waits an exponential time of rate λ_i, at the end of which it chooses one of its two neighbors with equal probability. If the chosen site is empty, the particle jumps to that site; if the site is occupied by another particle, the jump is not realized and a new exponential time starts afresh. We call this interaction the *exclusion* rule. This dynamics corresponds to a Markov process defined on the state space $\Omega_0^{ex} = \{\mathbf{x} \in \mathbb{Z}^{\mathbb{Z}}; x_i \neq x_j \text{ for } i \neq j\}$ and generated by the operator

$$
L_0^{ex} f(\mathbf{x}) = \frac{1}{2} \sum_{i \in \mathbb{Z}} \lambda_i \Big\{ \mathbf{1}(\mathbf{x} + e_i \in \Omega_0^{ex}) [f(\mathbf{x} + e_i) - f(\mathbf{x})]
$$
$$
+ \mathbf{1}(\mathbf{x} - e_i \in \Omega_0^{ex}) [f(\mathbf{x} - e_i) - f(\mathbf{x})] \Big\},
$$

where $\{e_i\}_i$ is the canonical basis in $\mathbb{Z}^{\mathbb{Z}}$. The number λ_i is interpreted as the *diffusion coefficient* of the particle x_i. We call the system $\{x_i(t)\}_i$ described in this way, the simple exclusion process with variable diffusion coefficient, and we call $\{\lambda_i\}_i$ the diffusion coefficient of

$\{x_i(t)\}$. Notice that when $\lambda_i = \lambda$ does not depend on i, the process $\{x_i(t)\}$ corresponds to a labeling of the usual simple exclusion process. We refer to this situation as the homogeneous case. It will be of interest to define the particle configuration $\eta_t \in \Omega^{ex} = \{0, 1\}^{\mathbb{Z}}$ by $\eta_t(x) = 1$ if $x_i(t) = x$ for some $i \in \mathbb{Z}$. Notice that η_t is *not* a Markov process, since in this case particles are indistinguishable, and therefore we can not keep track of the different diffusion coefficients. It will be useful as well to define the *environment process* $\eta_t^{env}(x) = \eta_t(x + x_0(t))$. Differently from the process η_t, η_t^{env} is indeed a Markov process. In fact, in dimension $d = 1$, due to the exclusion rule and the nearest-neighbor jumps, the relative ordering of particles is preserved by the dynamics. From now on, we assume, relabeling the particles if needed, that $x_i(t) < x_{i+1}(t)$ for any $i \in \mathbb{Z}$. Therefore, in the process η_t^{env}, the i-th particle at the right of the origin has diffusion coefficient λ_i, and analogously for particles standing at the left of the origin.

Our aim is to study the asymptotic behavior of a tagged particle for this system. Let us assume that $x_0(0) = 0$, that is, the 0-th particle starts at the origin. In the homogeneous case, it is well known that the behavior of a tagged particle is sub-diffusive, with a scaling exponent equal to $1/4$ (instead of the diffusive exponent $1/2$). We will see that this is also the case for the exclusion process with variable diffusion coefficient. For simplicity, we will assume that the sequence $\{\lambda_i\}_i$ is an i.i.d sequence of random variables satisfying the ellipticity condition $\epsilon_0 \leq \lambda_i \leq \epsilon_0^{-1}$ for any $i \in \mathbb{Z}$. More general conditions on $\{\lambda_i\}_i$ will be discussed below.

Now we explain how do we construct the initial configuration for the process $\{x_i(t)\}$. Let $u_0 : \mathbb{R} \to [0, 1]$ be a continuous function. For each $n > 0$, we define the initial distribution μ_n as the product measure in Ω^{ex} with marginals

$$\mu_n\{\eta; \eta(x) = 1\} = \begin{cases} u_0(x/n), & x \neq 0, \\ 1, & x = 0. \end{cases}$$

We define $x_0(0) = 0$. For $i > 0$, we define $x_i(0)$ recursively by $x_i(0) = \inf\{x > x_{i-1}(0); \eta(x) = 1\}$. For $i < 0$, we adopt a similar procedure. We denote this random initial configuration by $\mathbf{x}_{u_0}^n$. Now we are ready to state our first result, which is a law of large numbers for $x_0(t)$. For technical reasons, we will restrict ourselves to the case on which $\lim_{x \to \pm\infty} u_0(x) = u_0(\pm\infty)$ both exist and are different from $0, 1$ and on which $\sup_x u_0(x) < 1$.

Theorem 2.1 *Let $\{x_i(t)\}_i$ be the exclusion process with variable diffusion coefficient $\{\lambda_i\}_i$ and starting from $\mathbf{x}_{u_0}^n$. For almost every realization of the environment $\{\lambda_i\}_i$,*

$$\lim_{n \to \infty} \frac{x_0(tn^2)}{n} = r(t)$$

in probability, where $r(t)$ is the solution of

$$\int_0^{r(t)} u(t, x)dx = -\int_0^t \partial_x u(s, 0)ds,$$

$u(t, x)$ is the solution of the hydrodynamic equation $\partial_t u = \lambda \partial_x^2 u$ with initial condition u_0 and $\lambda = (E[\lambda_i^{-1}])^{-1}$.

Notice in the previous theorem, the subdiffusive scaling, the dependence on the *effective* diffusion coefficient λ and the *quenched* statement about the random diffusion coefficients.

Once we have obtained a law of large numbers, a natural question is whether a central limit theorem holds.

Theorem 2.2 *Consider an initial flat profile $u_0(x) \equiv \varphi$. Let $\{x_i(t)\}$ be the exclusion process with variable diffusion coefficient $\{\lambda_i\}_i$ and starting from $\mathbf{x}_\varphi^n$. Then,*

$$\lim_{n \to \infty} \frac{1}{n^{1/2}} x_0(tn^2) = \sqrt{\frac{(1 - \varphi)\sqrt{\lambda}}{\varphi}} \, \mathcal{Z}_t,$$

where $\mathcal{Z}_t$ is a symmetric fractional Brownian motion of Hurst index $1/4$ and the convergence is in the sense of finite-dimensional distributions convergence.

Notice in the previous theorem the explicit dependence of the limiting process in λ and φ.

3 The Coupling Method

As we mentioned before, the environment process η_t^{env} is a Markov process, but with a complicated non-local dynamics due to the ordering of particles. In order to obtain a local dynamics, we define a new process ξ_t with state space $\Omega^{zr} = \mathbb{N}_0^{\mathbb{Z}}$ by taking $\xi_t(i) = x_{i+1}(t) - x_i(t)$. It is easy to verify that ξ_t is a Markov process, generated by the operator

$$L^{zr} f(\xi) = \sum_{i \in \mathbb{Z}} \lambda_i \left\{ g(\xi(i)) \left[f(\xi^{i,i+1}) - f(\xi) \right] + g\left(\xi(i+1)\right) \left[f(\xi^{i+1,i}) - f(\xi) \right] \right\},$$

where $g : \mathbb{N}_0 \to \mathbb{R}$ is defined by $g(0) = 0$, $g(i) = 1$ if $i \geq 1$, and $\xi^{i,j}$ is defined by

$$\xi^{i,j}(l) = \begin{cases} \xi(i) - 1, & l = i, \\ \xi(j) + 1, & l = j, \\ \xi(l), & l \neq i, j. \end{cases}$$

The process ξ_t is known in the literature as a *zero-range process with bond disorder*. Although this functional transformation of the simple exclusion into a zero-range process was already exploited by Kipnis [15] to obtain the scaling limit of a tagged particle in the asymmetric simple exclusion process, the mathematical tools in order to treat the zero-range process with bond disorder have been developed only recently [9, 13]. The dynamics of this model can be described as follows. At each bond $\langle i, i + 1 \rangle$ we attach a *conductance* λ_i, corresponding to a exponential clock of rate λ_i. Each time this clock rings, we choose one of the directions $i \to i + 1$ or $i + 1 \to i$ with equal probability. Then we displace a particle from the origin site to the destination site. If there is not a particle on the origin site at that time, nothing happens. We do this independently for any bond $\langle i, i + 1 \rangle$.

Notice that the flat initial profile φ induces an initial distribution of particles which is left invariant by the evolution of η_t^{env}. It also introduces, via the previous mapping, an invariant measure of product form for the evolution of ξ_t. In the latter, the density of particles is $(1 - \varphi)/\varphi$. Let us denote by $j_{i,i+1}(t)$ the total current of particles through the bond $\langle i, i + 1 \rangle$ for the process ξ_t. The key observation is that the position of the tagged particle $x_0(t)$ in the original model is exactly equal to the current through the bond $\langle -1, 0 \rangle$ in the model ξ_t. Therefore, Theorem 2.1 is consequence of the following result:

Theorem 3.1 *Under the assumptions of Theorem* 2.1*,*

$$\lim_{n\to\infty} \frac{1}{n} j_{-1,0}(tn^2) = -\int_0^t \lambda\Phi(v(s,0))\partial_x v(s,0)ds,$$

where $v(t,x)$ *is the solution of the Cauchy problem*

$$\begin{cases} \partial_t v(t,x) = \partial_x(\lambda\Phi(v(t,x))\partial_x v(t,x)), \\ v(0,x) = v_0(x), \end{cases} \tag{3.1}$$

the function $\Phi(v)$ *is given by* $\Phi(v) = 1/(1+v)^2$ *and the initial condition* $v_0(x)$ *is defined below.*

About the initial condition v_0, we notice that our choice of $\mathbf{x}_{u_0}^n$ induces a choice ξ_0^n in the initial distribution of ξ_t as well. Therefore, we define v_0 as the functional transformation of u_0 in the continuous limit. This is given by the relation

$$\int_0^{z+\int_0^z v(x)dx} u(y)dy = z.$$

Defining $V(z) = z + \int_0^z v_0(x)dx$, we see that V satisfies the ODE

$$\begin{cases} \frac{d}{dz}V(z) = u_0(V(z))^{-1}, \\ V(0) = 0, \end{cases}$$

from where we obtain v_0 by taking $v_0(x) = V'(x) - 1 = (1 - u_0(V(x)))/u_0(V(x))$. The same reasoning allows us to relate the densities u_t and v_t. See [21] for more details.

In the same way, Theorem 2.2 is consequence of the following result:

Theorem 3.2 *Under the assumptions of Theorem* 2.2*,*

$$\lim_{n\to\infty} \frac{1}{n^{1/2}} j_{-1,0}(tn^2) = \sqrt{\frac{(1-\varphi)\sqrt{\lambda}}{\varphi}} \mathcal{Z}_t.$$

3.1 The Law of Large Numbers

Equation (3.1) corresponds to the *hydrodynamic equation* of the process ξ_t. For the process ξ_t restricted to a finite volume, the so-called hydrodynamic limit has been obtained in [9] (see also [13]). Under the technical condition $u(\pm\infty) \neq 0, 1$, the result in [9] can be extended to infinite volume as in [4]. Therefore, we have the following

Proposition 3.3 *For any continuous function* $G : \mathbb{R} \to \mathbb{R}$ *of bounded support, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{x\in\mathbb{Z}} \xi_{tn^2}(x)G(x/n) = \int_{\mathbb{R}} G(x)v(t,x)dx$$

in probability with respect to the law of ξ_t *starting from* ξ_0^n*, where* $v(t,x)$ *is the solution of the hydrodynamic equation* (3.1)*.*

Notice that the particular form of the effective diffusion coefficient λ follows from the well known fact that in dimension 1, the homogenized diffusion coefficient is equal to the harmonic mean of the microscopic diffusion coefficient. See [13] for more details. Starting from this proposition, the law of large numbers for $j_{-1,0}(t)$ is easy to understand. In fact, the macroscopic density flux across 0 in the hydrodynamic equation is given by $J_0(t) = -\int_0^t \lambda \Phi(v(s,0))\partial_x v(s,0)ds$. Therefore, Theorem 3.1 states that the current $j_{-1,0}(t)$, when properly rescaled, converges to $J_0(t)$. Notice that Proposition 3.3 does not give the desired result immediately, since an exchange of limits between the derivative and the scaling has to be justified. In order to justify this exchange, we can proceed as in [12], so we will just give an outline of the arguments. When the total mass $\int v_0(x)dx$ of the initial condition $v_0(x)$ is finite, the integrated current satisfies

$$J_0(t) = \int_0^\infty \{v(t,x) - v(0,x)\}dx. \tag{3.2}$$

The identity is just saying that the total current across 0 is equal to the mass at time t to the right of the origin, minus the mass at the right of the origin at time 0, an integrated version of the conservation of mass. Let us define $G_0(x) = \mathbf{1}\{x \geq 0\}$, the Heaviside function. We can rewrite $J_0(t) = \int_{\mathbb{R}} G_0(x)(v(t,x) - v(0,x))ds$ and in an analogous way,

$$\frac{1}{n}j_{-1,0}(tn^2) = \frac{1}{n}\sum_{x \in \mathbb{Z}}\{\xi_{tn^2}(x) - \xi_0(x)\}G_0(x/n). \tag{3.3}$$

If the initial profile has non-null asymptotic densities $v_0(\pm\infty) = \lim_{x \to \pm\infty} v_0(x)$, then formula (3.2) still makes sense, but we can not send n to ∞ in (3.3), since G_0 is not of compact support. Let us define $G_l(x) = (1 - x/l)^+$ for $x \geq 0$ and $G_l(x) = 0$ for $x < 0$. Here $(\cdot)^+$ denotes positive part. Taking the Cauchy limit of the right side of the following expression, it is not difficult to see that

$$\frac{1}{n}j_{-1,0}(tn^2) = \lim_{l \to \infty}\frac{1}{n}\sum_{x \in \mathbb{Z}}\{\xi_{tn^2}(x) - \xi_0(x)\}G_l(x/l),$$

the limit being in $\mathcal{L}^2(\mathbb{P}_n)$, where $\mathbb{P}_n$ is the law of the process ξ_{tn^2} starting from ξ_0^n. Moreover, the limit is uniform in l. But now, since the support of G_l is compact, we can use the result of Proposition 3.3 to pass to the limit in the previous expression, and to obtain the desired result.

3.2 The Central Limit Theorem

We recall the strategy in order to prove a law of large numbers for the current $j_{-1,0}(t)$ followed in the previous section. The idea is to relate the current to some linear functionals of the density of particles. This relation holds for the microscopic model, and also for the density flux in the macroscopic equation. In this way, the scaling limit of the current should follow from the scaling limit of the density of particles. However, the functional considered was not continuous, and therefore a cut-off argument was needed in order to take the limit. The final step was to justify an exchange of limits between the scaling and the cut-off. This strategy also gives a central limit theorem for the current in the zero-range process ξ_t. Differently from [12], in the case of the zero-range process the central limit theorem for the density of particles has only been obtained in equilibrium, restricting ourselves to flat initial profiles.

Now we explain what do we mean by a central limit theorem for the density of particles. For the initial configuration $\mathbf{x}_\varphi^n$, the associated initial distribution ξ_0^n is a product, uniform measure $\bar{\nu}_\varphi$ in Ω^{zr} with marginals given by

$$\bar{\nu}_\varphi(\xi(x) = k) = \varphi(1 - \varphi)^k.$$

A simple computation shows that the measure $\bar{\nu}_\varphi$ satisfies detailed balance with respect to the evolution of ξ_t, and therefore it is an invariant, reversible measure for this dynamics. It is not hard to check that it is also an ergodic measure for ξ_t. Let us notice that the number of particles in the zero-range process ξ_t is locally conserved, and due to ergodicity, it is the only locally conserved quantity. Therefore, it is more natural to parametrize the family $\{\bar{\nu}_\varphi; \varphi \in [0, 1]\}$ of ergodic, invariant measures by the density of particles at each site. It is easy to see that the density of particles $\rho(\varphi)$ defined by $\rho(\varphi) = \int \xi(x)\bar{\nu}_\varphi$ satisfies $\rho(\varphi) = (1 - \varphi)/\varphi$. We define then $\varphi(\rho) = 1/(1 + \rho)$ and $\nu_\rho = \bar{\nu}_{\varphi(\rho)}$. Now fix a density $\rho \in (0, \infty)$ and consider the process ξ_t, starting from the measure ν_ρ. This is equivalent to start the exclusion process with variable diffusion coefficient from the configuration $\mathbf{x}_\varphi^n$ for $\varphi = 1/(1 + \rho)$.

Let $\mathcal{S}'(\mathbb{R})$ be the space of tempered distributions. Let us define the $\mathcal{S}'(\mathbb{R})$-valued process $\mathcal{Y}_t^n$ by

$$\mathcal{Y}_t^n(G) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} G(x/n)\big(\xi_{tn^2}(x) - \rho\big),$$

for G in $\mathcal{S}(\mathbb{R})$, the Schwartz space of test functions in $\mathbb{R}$. We call this process the *density fluctuation field*. The following proposition has been proved in [9]:

Proposition 3.4 *The fluctuation field $\mathcal{Y}_t^n$ converges in the sense of finite distributions to a generalized Ornstein-Uhlenbeck process of mean zero and characteristics $\lambda\Phi(\rho)\partial_x^2$ and $\sqrt{\lambda\rho/(1+\rho)}\partial_x$.*

As in the case of the law of large numbers, the current $j_{-1,0}(t)$ satisfies

$$\frac{1}{\sqrt{n}} j_{-1,0}\big(tn^2\big) = \lim_{l \to \infty} \mathcal{Y}_t^n(G_l) - \mathcal{Y}_0^n(G_l),$$

the limit being uniform in $\mathcal{L}^2(\mathbb{P}_n)$. Therefore, we conclude that

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} j_{-1,0}\big(tn^2\big) = \lim_{l \to \infty} \mathcal{Y}_t(G_l) - \mathcal{Y}_0(G_l).$$

This last expression is formally equivalent to $\mathcal{Y}_t(G_0) - \mathcal{Y}_0(G_0)$. The covariances of the process $\mathcal{Y}_t$ are given by

$$E\big[\mathcal{Y}_t(G)\mathcal{Y}_s(H)\big] = \rho(1 + \rho) \int G(x)S_{t-s}H(x)dx$$

for $t \geq s \geq 0$ and $G, H \in \mathcal{S}(\mathbb{R})$, where $S_t = e^{t\lambda\partial_x^2}$ is the heat semigroup. With the aid of this formula, it is not difficult to prove that the covariances of the process $\mathcal{Y}_t(G_0) - \mathcal{Y}_0(G_0)$ are the same of $\sqrt{\rho\sqrt{\lambda}}\mathcal{Z}_t$, which proves Theorem 3.2.

4 Conclusions and Remarks

1. Comparing our results with the classical result of Arratia [2] and the results in [13], we see that the variable diffusion coefficient enters into the scaling limit of the tagged particle only through the effective diffusion coefficient λ. Notice that this effective diffusion coefficient is the same obtained by homogenization, although our system is subdiffusive. This is due to the fact that, for particle systems with bond disorder, the disorder and the time evolution can be decoupled using a stochastic version of the compensated compactness lemma [9]. We conclude that the subdiffusive behavior is a purely dynamical feature, since the factor $\sqrt{(1-\varphi)\sqrt{\lambda}/\varphi}$ is the same appearing for a simple exclusion process with uniform diffusion coefficient λ. In particular, the variance of $x_0(t)$ is of order $\sqrt{2\lambda t/\pi}(1-\varphi)/\varphi$ as $t \to \infty$.

2. About our assumptions on the initial conditions for the law of large numbers, the product structure of the initial measure μ_n is far from necessary. The more general assumptions are the following. First we need the initial measure μ_n to be *associated* to some macroscopic profile u_0, in the sense that the particle density should converge in distribution to $u_0(\cdot)$ (for a precise definition, see Sect. 2.3 of [9]). The second assumption is a bound on the entropy density of the initial measure μ_n with respect to the measure μ^φ obtained as the product measure associated to the flat profile $u_0 \equiv \varphi$. We also need the distribution of ξ_0^n to be stochastically dominated by the measure ν_ρ for some $\rho > 0$. This imposes the restriction $\sup_x u_0(x) < 1$.

3. In the CLT for the tagged particle, a key input is the fluctuation result for the particle density, Proposition 3.4. A complete proof of the results in Proposition 3.4 is only available in equilibrium, making the choice of $\mathbf{x}_\varphi^n$ rather restrictive. To extend Proposition 3.4 to the non-equilibrium setting remains one of the main open problems in the theory of scaling limits of particle systems. Notice that such a result has been obtained in [13] for the simple exclusion process in random media and in [11] for the zero-range process. None of these approaches can be adapted to our situation, however.

4. The assumptions on the variable diffusion coefficients $\{\lambda_i\}_i$ are not, by far, optimal. We have taken $\{\lambda_i\}_i$ as an i.i.d. sequence just to fix ideas. In fact, in order to get the results of Theorem 2.1, it is enough that $\sup_i \lambda_i > 0$ and

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=-n}^{0}\lambda_i^{-1} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n}\lambda_i^{-1} = \lambda^{-1}$$

both exist and are different from 0. To obtain the results in Theorem 2.2, we need in addition the ellipticity condition $\epsilon_0^{-1} > \lambda_i > \epsilon_0$ for every i. In particular, both results remain true if for instance we take a periodic sequence $\{\lambda_i\}_i$ of diffusion coefficients.

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